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Broken Symmetries and Zero-Mass Bosons

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It is argued by reference to the example of quantum electrodynamics that zero-mass bosons are not necessarily present in a theory with broken symmetries.

I. INTRODUCTION

A THEORY, described by a Lagrangian which is invariant under a continuous group of transformations, may possess nonsymmetric solutions if the vacuum is not invariant, under the group. For example, Nambu and Jona-Lasinio[1] have shown how a finite fermion mass can arise from a formally γ_5 invariant theory. For such theories there exist general proofs that zero-mass bosons are necessarily present.[2,3] However, since these proofs involve operators whose matrix elements are ill defined in most theories,[4] it is necessary to look at each theory in detail to see if such zero bosons actually arise. This has been done by Nambu and Jona-Lasinio for their model. Let us review their argument.

They show that the homogeneous Bethe-Salpeter (B.S.) equation for a particle-antiparticle system has a zero-mass ($q^2=0$) solution in an approximation which is consistent with their equation for the fermion mass. Their theory was highly divergent and a cutoff was introduced in order to make the theory finite. This cutoff had the effect of making the above homogeneous Bethe-Salpeter equation into an eigenvalue problem to which the Fredholm theory was applicable. The existence of a solution to the homogeneous B.S. equation then implied that the corresponding inhomogeneous γ_5 vertex equation had no solution for $q^2=0$. For $q^2\neq 0$, the homogeneous B.S. equation had no solution and the inhomogeneous pseudoscalar vertex equation had a perfectly regular solution. Thus, in this case from the existence of a zero-mass solution to the homogeneous B.S. equation, one could conclude that the pseudoscalar vertex $\Gamma_5(q^2)$ had a pole at $q^2=0$.[4] This meant that a zero-mass pseudoscalar particle was present.

In this paper we would like to point out that the part of Nambu's argument which shows the existence of a zero-mass solution to the homogeneous B.S. equation is generalizable to any arbitrary γ_5 invariant theory. However, in this general situation, we cannot further conclude (as above) that there exists a zero-mass boson. We show this by studying the example of quantum electrodynamics with zero bare electron mass.[5] This theory is finite without a cutoff. In this case the homogeneous B.S. equation is not of the Fredholm type. Therefore, we are unable to carry through the above arguments to further conclude that $\Gamma_5(q^2)$ has a pole at $q^2=0$. Instead we find the approximate equation for $\Gamma_5(q^2)$ has no solution for any momentum q. Thus (in quantum electrodynamics) the existence of a solution to the homogeneous B.S. equation is related to the fact that the usually defined Γ_5 vertex does not exist. It tells us nothing about whether zero-mass bosons are actually present.

In Sec. II we present the B.S. equation in a form convenient for our discussion. In Sec. III, we present an argument of Goldstone which explicitly exhibits a solution to the exact zero-momentum B.S. equation for any γ_5 invariant theory with symmetry breaking solutions. In Sec. IV we show how a certain approximation in quantum electrodynamics provides a simple example of Goldstone's general argument. We then study in detail the Γ_5 vertex equation in the same approximation.

II. THE BETHE-SALPETER EQUATION

We will first write the equation for the two-body Green's function $F(xy;x'y')$ in a form appropriate to our discussion of the bound states of a fermion anti-

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[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).

[2] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[3] S. Bludman and A. Klein, Phys. Rev. **131**, 2364 (1963).

[4] Arguments involving formally conserved axial vector currents are often misleading. This is discussed separately by K. Johnson, Phys. Letters (to be published).

[5] K. Johnson, M. Baker, and R. Willey (to be published).

fermion system. $F(xy;x'y')$ is defined as

$$F(xy;x'y')=-\langle(\psi(x)\psi(y')\bar{\psi}(y)\bar{\psi}(x'))_+\rangle\epsilon(x,y',y,x')+G(xy)G(y'x'), \quad \text{(II.1)}$$

where

$$\epsilon(x,y',y,x')=\epsilon(xy')\epsilon(xy)\epsilon(xx')\epsilon(y'y)\epsilon(y'x')\epsilon(yx'),$$
$$\epsilon(xy)=(x_0-y_0)/|x_0-y_0|.$$

$G(xy)$ is the one-particle Green's function defined as

$$G(xy)=i\langle(\psi(x)\bar{\psi}(y))_+\rangle\epsilon(xy). \quad \text{(II.2)}$$

$G(xy)$ satisfies the equation

$$\{\gamma\cdot p+M[G]\}G=1, \quad \text{(II.3)}$$

where the mass operator $M[G]$ is a functional of G which is determined by the Lagrangian for the system under consideration. M is originally given explicitly in terms of not only G but also of higher order Green's functions which in turn are related to G by further equations. The solutions of these equations in terms of G then allows us to express M as a functional of G alone. G is then determined by solving Eq. (II.3). Usually we can only construct the perturbation expansion of $M[G]$. In this case $M[G]$ can be represented by the sum of all proper self-energy diagrams containing no fermion self-energy insertions. In each diagram an internal fermion line represents the full Green's function G, while all vertices and other internal lines represent bare vertices and propagators.

In the Appendix we show that F satisfies the Schwinger,[6] Bethe-Salpeter[7] integral equation

$$F(xy;x'y')=G(xx')G(yy')+\int d^4\xi d^4\eta d^4\xi' d^4\eta' G(x\xi)\times I(\xi\eta;\xi'\eta')F(\xi'\eta':x'y')G(\eta y), \quad \text{(II.4)}$$

where the interaction operator I is given as a functional of G by the equation[8]

$$I(\xi\eta;\xi'\eta')=-\delta M[\xi\eta;G]/\delta G(\xi'\eta'). \quad \text{(II.5)}$$

Equation (II.4) then becomes an explicit equation for F when use if made of Eq. (II.3) to determine G. Of course G must be left arbitrary until after the functional differentiation of (II.5) to determine I is carried out. In momentum space, Eq. (II.4) becomes

$$F(p,q;K)=(2\pi)^4\delta^4(p-q)G(p+\tfrac{1}{2}K)G(p-\tfrac{1}{2}K)+G(p+\tfrac{1}{2}K)\int\frac{d^4s}{(2\pi)^4}I(p,s;K)\times F(p,s;K)G(p-\tfrac{1}{2}K), \quad \text{(II.6)}$$

where

$$G(xx')=\int\frac{d^4p}{(2\pi)^4}G(p)e^{ip\cdot(x-x')},$$

$$F(xy;x'y')=\int\frac{d^4K}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}F(p,q;K)\times\exp\{K\cdot[\tfrac{1}{2}(x+y)-\tfrac{1}{2}(x'+y')]+p\cdot(x-y)-q(x'-y')\} \quad \text{(II.7)}$$

and likewise for I.

If there exists a bound state of a fermion-antifermion system of mass $K^2=-M_B^2$, it can be shown that[9]

$$F(p,q;K)\xrightarrow[K^2\to -M_B^2]{}\frac{\psi(p,K)\psi^\dagger(q,K)}{K^2+M_B^2},$$

where $\psi(p,K)$ satisfies the homogeneous Bethe-Salpeter equation

$$\psi(p,K)=G(p+\tfrac{1}{2}K)\int\frac{d^4s}{(2\pi)^4}I(p,s;K)\times\psi(s,K)G(p-\tfrac{1}{2}K), \quad \text{(II.8)}$$

with $K^2=-M_B^2$. Whether or not the existence of a solution to Eq. (II.8) implies that the theory contains a bound state depends upon the detailed nature of Eq. (II.8).[10] This will be discussed in Sec. IV with reference to the case of quantum electrodynamics.

III. SOLUTION OF THE B.S. EQUATION IN A SYMMETRY BREAKING THEORY[11]

We assume our theory is described a Lagrangian $\mathfrak{L}$ which is invariant under the γ_5 gauge transformation

$$\psi\to e^{i\gamma_5\theta/2}\psi,\quad \bar{\psi}\to\bar{\psi}e^{i\gamma_5\theta/2}. \quad \text{(III.1)}$$

We write

$$G(p)=G_1(p^2)+\gamma\cdot pG_2(p_2). \quad \text{(III.2)}$$

Now if the vacuum is invariant under the unitary operator which induces the transformation of Eq. (III.1), then it follows that

$$G(p)=e^{i\gamma_5\theta/2}G(p)e^{i\gamma_5\theta/2}, \quad \text{(III.3)}$$

from which we conclude

$$G_1(p^2)=0. \quad \text{(III.4a)}$$

[6] J. Schwinger, Proc. Natl. Acad. Sci. **37**, 452, 456 (1951).
[7] E. E. Salpeter and H. A. Bethe, Phys. Rev. **87**, 1232 (1951).
[8] This result has also been noted by G. Baym, Phys. Letters **1**, 241 (1962) in the context of many-body problems.
[9] See, for instance, S. Mandelstam, Proc. Roy. Soc. (London) **237**, 496 (1956).
[10] For example, when the Fredholm theory is applicable to Eq. (II.6), the existence of a solution to Eq. (II.8) implies a bound state. Nambu's model (Ref. 1) is of this type due to the cutoff procedure used.
[11] In this section, we make use of an argument originally due to J. Goldstone (private communication to M. Baker). We wish to thank Dr. J. Goldstone for communicating the argument of this section to one of us.

If the vacuum is not invariant then Eq. (II.3) may have a solution G for which

$$G_1(p^2)\neq 0. \qquad \text{(III.4b)}$$

In the following discussion we allow for either possibility to occur. However, we assume that $M[G]$ is given by its usual functional expansion in terms of G (or by any other approximation which preserves the γ_5 symmetry of the perturbation expansion). It then follows

$$M[e^{i\gamma_5\theta/2}Ge^{i\gamma_5\theta/2}]=e^{-i\gamma_5\theta/2}M[G]e^{-i\gamma_5\theta/2}. \qquad \text{(III.5)}$$

The differential version of Eq. (III.5) yields;

$$\{\gamma_5,M\}=-(\delta M/\delta G)\{\gamma_5,G\}, \qquad \text{(III.6)}$$

where $\{A,B\}=AB+BA$.

Noting that

$$\{\gamma_5,M\}=G^{-1}\{\gamma_5,G\}G^{-1},$$

we write Eq. (III.6) in coordinate space

$$\{\gamma_5,G(xy)\}=-\int d^4x'd^4y'd^4\xi d^4\eta G(xx')\frac{\delta M(x'y')}{\delta G(\xi\eta)} \times\{\gamma_5,G(\xi\eta)\}G(y'y). \qquad \text{(III.7)}$$

Transforming Eq. (III.7) into momentum space and using (II.5) we obtain

$$\{\gamma_5,G(p)\}=G(p)\int\frac{d^4q}{(2\pi)^4}I(p,q;0)\{\gamma_5,G(q)\}G(p) \qquad \text{(III.8a)}$$

or

$$\{\gamma_5,M(p)\}=\int\frac{d^4q}{(2\pi)^4}I(p,q;0)G(q) \times\{\gamma_5,M(q)\}G(q). \qquad \text{(III.8b)}$$

Since $\{\gamma_5,G(p)\}=2\gamma_5G_1(p^2)$, Eqs. (III.8) are nontrivial only if $G_1(p^2)\neq 0$. In that case we conclude that

$$\psi(p,0)=\gamma_5G_1(p^2) \qquad \text{(III.9)}$$

is solution of the homogeneous Bethe-Salpeter Eq. (II.8) for $K=0$.

In all cases that we are interested in Eq. (II.8) can be transformed into an equation in the Euclidean space.[12] This transformation will be carried out in Sec. IV in the case of quantum electrodynamics. If we assume this can be done, then it follows that Eq. (III.9) is also a solution of Eq. (II.8) when $K^2=0$.[12]

However, this is all that can be concluded from the γ_5 invariance of our theory on the basis of general arguments. In Nambu's model, from the existence of a solution of Eq. (II.8) for $K^2=0$, one could conclude that the theory contained a zero-mass pseudoscalar boson. In the next section we shall show that this is not the case for quantum electrodynamics.

[12] While a pseudoscalar solution to Eq. (II.8) for $K^2=0$ satisfies Eq. (II.8) for $K=0$, the converse is not necessarily true, unless, for example, Wick's rotation of the p_0-contour is possible [G. C. Wick, Phys. Rev. **96**, 1124 (1954)]. A solution which satisfies Eq. (II.8) for $K=0$, but not for $K^2=0$, $|\mathbf{K}|=K_0\neq 0$ would represent a "spurion" wave function, transforming like the vacuum under the Lorentz transformations. We are grateful to Professor O. W. Greenberg and Professor A. Wightman for pointing out this possibility.

IV. QUANTUM ELECTRODYNAMICS

We now study the example of quantum electrodynamics with zero-bare electron mass[5] in the approximation where M is given by[13]

$$M[p;G]=ie_0^2\int\frac{d^4q}{(2\pi)^4}\mathfrak{D}_{\mu\nu}(p-q)\gamma^\mu G(q)\gamma^\nu \qquad \text{(IV.1)}$$

with

$$\mathfrak{D}_{\mu\nu}(q)=(g_{\mu\nu}-q_\mu q_\nu/q^2)(q^2-i\epsilon)^{-1}.$$

Inserting Eq. (IV.1) into Eq. (II.3) we get an approximate integral equation for G:

$$G(p)^{-1}=\gamma\cdot p+ie_0^2\int\frac{d^4q}{(2\pi)^4}\mathfrak{D}_{\mu\nu}(p-q)\gamma^\mu G(q)\gamma^\nu. \qquad \text{(IV.2)}$$

It is argued in Ref. 5 that Eq. (IV.2) gives the asymptotic behavior of $G(p)$ for large p of the full theory. Now as most of the discussion of this section depends only upon the asymptotic properties of $G(p)$, some of the resulting conclusions may be valid independent of the particular approximation of Eq. (IV.1).

In order to avoid an infrared divergence in Eq. (IV.2), we must give the photon a small mass. This will not effect any of the asymptotic properties of Eq. (IV.2), but will only modify it in the small p region where Eq. (IV.2) is not expected to be a good approximation to the full theory.

We now can break up Eq. (IV.2) into two coupled equations for $G_1(p^2)$ and $G_2(p^2)$

$$-\frac{G_2(p^2)\gamma\cdot p}{G_1^2(p^2)+p^2G_2^2(p^2)} =\gamma\cdot p+ie_0^2\int\frac{d^4q}{(2\pi)^4}\mathfrak{D}_{\mu\nu}(p-q)\gamma^\mu G_2(q^2)\gamma\cdot q\gamma^\nu, \qquad \text{(IV.3a)}$$

$$\frac{G_1(p^2)}{G_1^2(p^2)+p^2G_2^2(p^2)} =ie_0^2\int\frac{d^4q}{(2\pi)^4}\mathfrak{D}_{\mu\nu}(p-q)\gamma^\mu G_1(q^2)\gamma^\nu. \qquad \text{(IV.3b)}$$

The condition that electron has finite mass m requires that

$$G(p)^{-1}|_{\gamma\cdot p=-m}=0. \qquad \text{(IV.4)}$$

In Ref. 5, finite solutions of Eq. (IV.2) subject to condition (IV.4) have been found for any value of

[13] In addition to Eq. (IV.1) there is a second contribution to M which is linear in G. This is the diagram where a photon is emitted from the electron and gets absorbed in a closed loop ("tadpole" diagram). Such a term of course vanished in electrodynamics (cf., Ref. 1). However, when differentiated according to Eq. (II.5) it produces the annihilation graph contribution to I. We have omitted this term in I since it does not contribute anything to the pseudoscalar wave function or vertex.

$m\neq 0$. The value of m just serves as a scale for the solution. Asymptotically the solution behaves like

$$\lim_{p\to\infty} G(p)^{-1}=\gamma\cdot p+\mathcal{O}[p^{-1+(1-3\alpha_0/\pi)^{\frac{1}{2}}}]. \quad \text{(IV.5)}$$

From the discussion of Sec. III we should expect that $G_1(p^2)\gamma_5$ should be the solution of the $K=0$ B.S. equation with the interaction operator I calculated from (IV.1) according to Eq. (II.5). In this approximation Eq. (II.8) becomes

$$\psi(p,K)=-ie_0^2G(p+\tfrac{1}{2}K)\times\int\frac{d^4s}{(2\pi)^4}\mathfrak{D}_{\mu\nu}(p-s)\gamma^\mu\psi(s,K)\gamma^\nu G(p-\tfrac{1}{2}K). \quad \text{(IV.6)}$$

If we now put $K=0$ in Eq. (IV.6) and set $\psi(p,0)=\gamma_5G_1(p^2)$ we get back precisely Eq. (IV.3b).

This example should give us a clear picture of what is happening in the general situation described by Eq. (III.8a). There, by virtue of the γ_5, invariance Eq. (III.8a) reduces to an equation for $G_1(p^2)$ which is automatically satisfied if G is a solution of Eq. (II.3). Thus, Eq. (III.8a) can be understood as being just an alternate way of writing the homogeneous part of Eq. (II.3) in a γ_5 invariant theory. This is all that we can conclude on grounds of the γ_5 invariances alone.

We shall now show that the existence of a solution to the homogeneous Eq. (IV.6) for $K=0$ has nothing to do with possible presence of massless pseudoscalar mesons. Such mesons would produce poles not only in F, but also in the Γ_5 vertex operator which in the approximation (IV.1), satisfies the inhomogeneous equation

$$\Gamma_5(p+q,p)=\gamma_5-ie_0^2\int\frac{d^4p'}{(2\pi)^4}\mathfrak{D}_{\mu\nu}(p-p')\gamma^\mu G(p'+q)\times\Gamma_5(p'+q,p')G(p')\gamma^\nu. \quad \text{(IV.7)}$$

The homogeneous version of Eq. (IV.7) with a possible solution $\Gamma_5^H(p+q,p)$ is identical to (IV.6) with the identification

$$G(p+q)\Gamma_5^H(p+q,p)G(p)=\psi(p+q/2;q). \quad \text{(IV.8)}$$

Therefore, the existence of a solution to the homogeneous B.S. Eq. (IV.6) can equivalently be considered as a necessary condition for the existence of a pole in $\Gamma_5(p+q,p)$. Since we have explicitly constructed a solution to Eq. (IV.6) for $K=0$ or $q=0$, there is a possibility that $\Gamma_5(p+q,p)$ has a pole at $q^2=0$. We will show below that this does not happen. Instead we shall show without recourse to perturbation theory that the Γ_5 vertex does not exist for any q in the approximation (IV.7). To do this we rewrite Eq. (IV.7) in the following way:

$$\Gamma_5(p+q,p)=I_5(p+q,p)-ie_0^2\int\frac{d^4s}{(2\pi)^4}\mathfrak{D}_{\mu\nu}(p-s)\gamma^\mu G(s)\times\Gamma_5(s+q,s)G(s)\gamma^\nu, \quad \text{(IV.8a)}$$

where

$$I_5(p+q,p)=\gamma_5-ie_0^2\int\frac{d^4s}{(2\pi)^4}\mathfrak{D}_{\mu\nu}(p-s)\gamma^\mu\times[G(s+q)-G(s)]\Gamma_5(s+q,s)G(s)\gamma^\nu. \quad \text{(IV.8b)}$$

Let us suppose that a solution of Eq. (IV.7) for $\Gamma_5(p+q,q)$ exists for a particular value of q. Then $\Gamma_5(p'+q,p')$ must fall off rapidly enough for large p' to ensure the convergence of the integral in Eq. (IV.7). But if this is so, then the integral on the right-hand side of Eq. (IV.8b) converges so rapidly that $I_5(p+q,p)\to\gamma_5$ as $p\to\infty$. We will show below that if

$$I_5(p+q,p)\to\gamma_5 \quad \text{as} \quad p\to\infty, \quad \text{(IV.9)}$$

then Eq. (IV.8a) has no solutions. Thus, the original assumption that Eq. (IV.7) has a solution leads to a contradiction.

In order to show that Eq. (IV.8a), has no solutions if I_5 behaves according to (IV.9), it will be sufficient to study the high p limit of Eq. (IV.8a). We can then neglect the photon mass which is present in $\mathfrak{D}_{\mu\nu}(p-p')$. We write

$$I_5(p+q,p)=\gamma_5I(p^2,q^2,p\cdot q), \quad \text{(IV.10)}$$

where

$$I(p^2,q^2,p\cdot q)\to 1, \quad \text{as} \quad p\to\infty. \quad \text{(IV.11)}$$

Other invariants occur in I_5, but they do not effect the high p limit of Eq. (IV.8a). We set

$$\Gamma_5(p+q,p)=\gamma_5\Gamma(p^2,q^2,p\cdot q). \quad \text{(IV.12)}$$

Equation (IV.8a) then becomes

$$\Gamma(p^2,q^2,p\cdot q)=I(p^2,q^2,p\cdot q)-3ie_0^2\int\frac{d^4s}{(2\pi)^4}\frac{\Gamma(s^2,q^2,s\cdot q)}{(p-s)^2-i\epsilon}\times[G_1^2(s^2)+s^2G_2^2(s^2)]. \quad \text{(IV.13)}$$

We can analytically continue (IV.13) to the Euclidean region $p^2>0$, $q^2>0$, $-1\leq\cos\theta\equiv(p\cdot q/|p||q|)\leq 1$. We then expand $\Gamma(p^2,q^2,p\cdot q)$ in terms of the Tschebyscheff polynormals[14] $C_n(\cos\theta)$:

$$C_n(\cos\theta)=[\sin(n+1)\theta/\sin\theta]. \quad \text{(IV.14)}$$

$$\Gamma(p^2,q^2,p\cdot q)=\sum_{n=0}^{\infty}C_n(\cos\theta)\Gamma_n(p^2,q^2). \quad \text{(IV.15)}$$

Equation (IV.13) then becomes

$$\Gamma_n(p^2,q^2)=I_n(p^2,q^2)+\frac{3e_0^2}{8\pi^2(n+1)}\left\{\frac{1}{p^{n+1}}\int_0^p dp'p'^2\times[G_1^2(p'^2)+p'^2G_2^2(p'^2)]p'^{n+1}\Gamma_n(p'^2,q^2)+p^n\int_p^\infty\frac{dp'p'^2}{p'^{n+1}}[G_1^2(p'^2)+p'^2G_2^2(p'^2)]\times\Gamma_n(p'^2,q^2)\right\}, \quad \text{(IV.16)}$$

[14] See, for example, M. Baker and I. Muzinich, Phys. Rev. **132**, 2291 (1963).

where

$$I_n(p^2,q^2)=\frac{2}{\pi}\int_0^\pi d\theta\,\sin^2\theta C_n(\cos\theta)I(p^2,q^2,pq\cos\theta)\,. \quad \text{(IV.17)}$$

Now according to Eq. (IV.4)

$$p^2[G_1{}^2(p^2)+p^2G_2{}^2(p^2)]\rightarrow 1, \quad \text{as} \quad p\rightarrow\infty\,. \quad \text{(IV.18)}$$

Also, from Eqs. (IV.11) and (IV.17), we find

$$\lim_{p^2\to\infty} I_n(p^2,q^2)=\delta_{n,0}\,. \quad \text{(IV.19)}$$

Hence, (IV.17) becomes in the limit of large p

$$\Gamma_n(p^2,q^2)\simeq\delta_{n,0}+\frac{3e_0{}^2}{8\pi^2(n+1)}\left\{\frac{1}{p^{n+2}}\int^p dp'p'^{n+1}\Gamma_n(p'^2,q^2)\right.$$
$$\left.+p^n\int_p^\infty dp'p'^{-n-1}\Gamma_n(p'^2,q^2)\right\}. \quad \text{(IV.20)}$$

Equation (IV.20) is readily converted into a second-order homogeneous equation with the general solution

$$\lim_{p\to\infty}\Gamma_n(p^2,q^2)=C_1p^{-1+[(n+1)^2-3\alpha_0/\pi]^{1/2}}$$
$$+C_2p^{-1-[(n+1)^2-3\alpha_0/\pi]^{1/2}}, \quad \text{(IV.21)}$$

where $\alpha_0=e_0{}^2/4\pi$.

The $\delta_{n,0}$ term in Eq. (IV.20) did not contribute to the derived differential equation. It does impose a boundary condition upon the solution (IV.21) which is clearly not satisfied for the $n=0$ equation. For when (IV.21) is substituted into (IV.20), the left-hand side vanishes for large p for all q^2 while the left-hand side approaches $\delta_{n,0}$. Hence, Eq. (IV.16) has no solution for $n=0$. For $n\neq0$ there is a solution of the asymptotic Eq. (IV.20) for all q^2.

Using Eq. (IV.14) we then conclude that Eq. (IV.13) has no solution if $I\rightarrow1$ as $p\rightarrow\infty$. Hence, there are no solution to our original Eq. (IV.7) for any q. One might have argued that this result is obvious since the perturbation expansion of Eq. (IV.7) leads to divergent integrals. However, if one had restricted himself to using perturbation theory, one would also conclude that there were no solutions to (IV.3b), i.e., $G_1(p^2)=0$. It is therefore essential to carry out arguments without recourse to perturbation theory. To repeat: We have obtained a symmetry breaking solution $G_1(p^2)\neq0$ without necessarily having a zero-mass pseudoscalar boson in the same approximation.

V. CONCLUSION

We have explicitly exhibited a solution to the zero-momentum homogeneous Bethe-Salpeter equation for a theory with a broken invariance. In the case of quantum electrodynamics the existence of this solution is symptomatic not of the presence of zero-mass particles but rather of the nonexistence of the usually defined vertex operator. We conclude that in order to determine whether zero-mass particles are actually present in any theory with a broken symmetry, one must investigate that theory in detail. In particular, the distinction between theories which are finite without cutoff and those which are cutoff dependent is essential.

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APPENDIX

In order to derive Eq. (II.5), it is convenient to introduce external sources η and $\bar{\eta}$ which anticommute with themselves and all fermion operators. In the presence of such sources the Lagrangian is

$$\mathcal{L}[x,\eta,\bar{\eta}]=\mathcal{L}(x)+\tfrac{1}{2}[\bar{\eta}(x),\psi(x)]+\tfrac{1}{2}[\bar{\psi}(x),\eta(x)], \quad \text{(A1)}$$

where $\mathcal{L}(x)$ is the Lagrangian in the absence of sources.

If we define the one-particle Green's function in the presence of sources as

$$G(xx')=i\langle(\psi(x)\bar{\psi}(x'))_+\rangle_{\eta\bar{\eta}}\epsilon(xx')/\langle0|0\rangle_{\eta\bar{\eta}}. \quad \text{(A2)}$$

Then $F(xy;x'y')$ of Eq. (II.1) can be expressed as

$$F(xy,x'y')=i\left.\frac{\delta^2G(xy)}{\delta\bar{\eta}(y')\delta\eta(x')}\right|_{\eta=\bar{\eta}=0}. \quad \text{(A3)}$$

Now we define the mass operator M in the presence of external source by the equation

$$G=(\gamma\cdot p+M)^{-1}+i(\gamma\cdot p+M)^{-1}\eta\bar{\eta}(\gamma\cdot p+M)^{-1}. \quad \text{(A4)}$$

[The second term in (A4) represents the disconnected diagrams.] From Eqs. (A3) and (A4) we get

$$F=GG-iG(\delta^2M/\delta\bar{\eta}\delta\eta)_{\eta=\bar{\eta}=0}G. \quad \text{(A5)}$$

If now M is expressed in terms of G as discussed in II, the dependence of M on η and $\bar{\eta}$ can be expressed completely by its dependence upon G. Thus,

$$\left.\frac{\delta^2M}{\delta\bar{\eta}\delta\eta}\right|_{\eta=\bar{\eta}=0}=\left(\frac{\delta M}{\delta G}\right)\left.\frac{\delta^2G}{\delta\bar{\eta}\delta\eta}\right|_{\eta=\bar{\eta}=0}, \quad \text{(A6)}$$

whence (A5) becomes

$$F=GG-G(\delta M/\delta G)FG, \quad \text{(A7)}$$

which is the desired results Eqs. (II.4) and (II.5).